RELATIONS ON SPACES

By

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CHAPTER I

PRELIMINARIES

A relation R *on a topological space* X *is a subset of* $X \times X$. If A is a subset of X and if R is a relation on X, let

$$AR = p_2\ ((A \times X) \cap R)$$

and

$$RA = p_1\ ((X \times A) \cap R)$$

where p_1, p_2 are respectively the first and the second projections. For x in X we shall write simply xR for $\{ x \} R$.

Let

$$R^{(-1)} = \{ (y, x) \mid (x, y) \in R \} ,$$

$$AR^{[-1]} = \{ x \mid xR \subset A \} ,$$

and

$$R^{[-1]}A = \{ y \mid Ry \subset A \} .$$

In all that follows, X *will always denote a Hausdorff space, and* R *a relation on* X. The closure of a set A will be denoted by A^* and the interior of A by A° .

Definition 1.1. A relation R on X is said to be

$$\left\{\begin{array}{l}\underline{\text{lower semi-continuous}}\ \text{(abbreviated l.s.c.)}\\ \\ \underline{\text{upper semi-continuous}}\ \text{(abbreviated u.s.c.)}\end{array}\right\}$$

at x in X if and only if x in $\left\{\begin{matrix} VR^{(-1)} \\ VR^{[-1]} \end{matrix}\right\}$ implies x in $\left\{\begin{matrix} (VR^{(-1)})^{\circ} \\ (VR^{[-1]})^{\circ} \end{matrix}\right\}$ for all $V = V^{\circ} \subset X$. R is said to be $\left\{\begin{matrix} \text{l.s.c.} \\ \text{u.s.c.} \end{matrix}\right\}$ on X if and only if R is $\left\{\begin{matrix} \text{l.s.c.} \\ \text{u.s.c.} \end{matrix}\right\}$ at x for all x in X. R is said to be *continuous* if and only if R is both l.s.c. and u.s.c.

Definition 1.2. A relation R on X is $\left\{\begin{matrix} \underline{\text{upper semiclosed}}\ \text{(u.s.cl.)} \\ \underline{\text{lower semiclosed}}\ \text{(l.s.cl.)} \end{matrix}\right\}$ if and only if $\left\{\begin{matrix} xR \\ Rx \end{matrix}\right\}$ is closed for any x in X.

Definition 1.3. A relation R on X is said to be a *partial order*[1] if and only if the following conditions are satisfied:

(a) (reflexivity): $(x, x) \in R$ for all x in X,

(b) (antisymmetry): $(x, y) \in R$ and $(y, x) \in R$ imply $x = y$,

(c) (transitivity): $(x, y) \in R$ and $(y, z) \in R$ imply $(x, z) \in R$.

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[1] A relation R is said to be a *quasi-order* if (a) and (c) are satisfied.

We here state some theorems to which we will refer throughout this work. The proofs of these theorems may be found in [19].

Theorem 1.1. If R is closed in $X \times X$ and if A is compact, then both AR and RA are closed; moreover, if V is an open set then $R^{[-1]} V$ and $VR^{[-1]}$ are open.

Theorem 1.2. The following statements are equivalent:

(a) R is l.s.c. on X.

(b) $RA^{\circ} \subset (RA)^{\circ}$ for all $A \subset X$.

(c) $A^{*} R \subset (AR)^{*}$ for all $A \subset X$.

(d) RV is open for all open $V \subset X$.

(e) $p_1 \mid R$ is open, where $p_1 \mid R$ is the restriction of p_1 to R.

Theorem 1.3. The following statements are equivalent:

(a) R is u.s.c.

(b) $VR^{[-1]}$ is open for all $V \subset X$.

(c) $AR^{(-1)}$ is closed for all closed $A \subset X$.

(d) $A^{\circ} R^{[-1]} \subset (AR^{[-1]})^{\circ}$ for any $A \subset X$.

(e) $(AR^{(-1)})^{*} \subset A^{*} R^{(-1)}$ for any $A \subset X$.

Theorem 1.4. If X is a compact Hausdorff space, then $R = R^*$ if and only if

(a) R is u.s.c. on X, and

(b) R is u.s.cl. on X.

Definition 1.4. A subset C of X is an R-chain if and only if $C \times C \subset R \cup R^{(-1)}$.

As a consequence of the well-known Hausdorff maximality principle, there is a maximal R-chain for any relation R on any space X. A useful result of Wallace [14] is the following.

Theorem 1.5. If R is both u.s.cl. and l.s.cl. quasi-order, then every maximal R-chain is closed.

Definition 1.5. An element a in X is R-minimal (R-maximal), whenever $(x, a) \in R$ $((a, x) \in R)$ implies $(a, x) \in R$ $((x, a) \in R)$.

The following fundamental theorem first proved in [14] will be used repeatedly.

Theorem 1.6. If R is a u.s.cl. (l.s.cl.) relation on a compact space X, and if A is a non-void closed subset of X, then A contains an R-maximal (R-minimal) element a of A.

The Alexander-Kolmogoroff cohomology groups will be used as developed in [21]. In what follows the coefficient group is fixed and therefore will not be mentioned. We record here some useful theorems from [21].

Theorem 1.7. If $B \subset A \subset X$, then

$$\begin{array}{l} H^{o}(X,A) \xrightarrow{j^*} H^{o}(X,B) \xrightarrow{i^*} H^{o}(A,B) \xrightarrow{\delta} \\ H^{1}(X,A) \xrightarrow{j^*} H^{1}(X,B) \xrightarrow{i^*} H^{1}(A,B) \xrightarrow{\delta} \\ \text{-----------------------------} \xrightarrow{\delta} \\ H^{n}(X,A) \xrightarrow{j^*} \text{-----------------} \end{array}$$

is an exact sequence, where i^* and j^* are the induced homomorphisms of the inclusion maps i and j from A to X and X to X, respectively, and δ is the coboundary operator for the triple (X, A, B).

Theorem 1.8. For the space X and any connected set A of X the homomorphism i^* of

$$H^{o}(X) \xrightarrow{i^*} H^{o}(A) \xrightarrow{\delta} H^{1}(X, A)$$

is an epimorphism and hence δ is 0.

For simplicity we denote by $f:(X, A) \to (Y, B)$ *that* f *is a function from* X *to* Y *and* A *is a subset of* X *such that* $f(A) \subset B \subset Y$. *Furthermore, if* $i:(X, A) \to (X, B)$ *is an inclusion map, we will write*

$i:(X, A) \subset (X, B)$.

Theorem 1.9. If $f:(X, A, B) \to (X', A', B')$ is continuous, and if $u: (X, A) \to (X', A')$,

$v: (X, B) \to (X', B')$,

$w: (A, B) \to (A', B')$

are defined by

$$u(x) = v(x) = w(x) = f(x),$$

then the ladder

$$\begin{tikzcd}
{} \arrow[r, "\delta"] & H^p(X, A) \arrow[r, "j^*"] \arrow[d, "u^*"] & H^p(X, B) \arrow[r, "i^*"] \arrow[d, "v^*"] & H^p(A, B) \arrow[r, "\delta"] \arrow[d, "w^*"] & {} \\
{} \arrow[r, "\delta"'] & H^p(X, A) \arrow[r, "j^*"'] & H^p(X, B) \arrow[r, "i^*"'] & H^p(A, B) \arrow[r, "\delta"'] & {}
\end{tikzcd}$$

is analytic, that is, each rectangle of the ladder is analytic.

We state here a special case of The Mayer-Vietoris Sequence which will be sufficient in what follows for our purpose.

Theorem 1.10. If X is a compact Hausdorff space, and if $X = X_1 \cup X_2$ where X_1 and X_2 are closed subsets of X, then there exists an exact sequence

$$\xrightarrow{I^*} H^{q-1}(X_1 \cap X_2) \xrightarrow{\Delta} H^q(X) \xrightarrow{J^*} H^q(X_1) \times H^q(X_2) \xrightarrow{I^*}$$

$$H^q(X_1 \cap X_2) \xrightarrow{\Delta} \cdots$$

where $(j^*_\alpha, i^*_\alpha, \delta_\alpha)$ and $(\bar{j}^*_\alpha, \bar{i}^*_\alpha, \bar{\delta}_\alpha)$ are the homomorphisms in the exact sequences for the triples $(X, X_\alpha, \square)$ and $(X_\alpha, X_1 \cap X_2, \square)$ respectively for $\alpha = 1, 2$;
$k: (X_2, X_1 \cap X_2) \subset (X, X_1)$; $J^* = j^*_1 \times j^*_2$;
$I^* = i^*_1 - i^*_2$; and $\Delta = j^*_1 k^{*-1} \bar{\delta}_2$.

As a consequence of the foregoing theorem we have the following

Theorem 1.11. With the hypotheses of Theorem 1.10 and if $X_1 \cap X_2$ is connected, the homomorphism Δ in

$$H^0(X_1 \cap X_2) \xrightarrow{\Delta} H^1(X) \xrightarrow{J^*} H^1(X_1) \times H^1(X_2) \xrightarrow{I^*}$$

is 0.

Proof. Observe the following ladder

$$\begin{tikzcd}
H^0(X) \arrow[r, "\delta_1"] \arrow[d, "u^*"] & H^1(X, X_1) \arrow[r, "j^*_1"] \arrow[d, "k^*"] & H^1(X) \arrow[r, "i^*_1"] & {} \\
H^0(X_1 \cap X_2) \arrow[r, "\bar{\delta}_2"] & H^1(X_2, X_1 \cap X_2) \arrow[r, "\bar{j}^*_2"] & H^1(X_2) \arrow[r, "\bar{i}^*_2"] & {}
\end{tikzcd}$$

where $u: X_1 \cap X_2 \subset X$ and where (j^*_1, i^*_1, δ_1), $(\bar{j}^*_2, \bar{i}^*_2, \bar{\delta}_2)$ and k are as in Theorem 1.10. It follows from Theorem 1.8 that if h is in $H^0(X_1 \cap X_2)$ then there is an h' in $H^0(X)$ such that $u^*(h') = h$. Thus by Theorem 1.9., we have

$$k^{*^{-1}} \bar{\delta}_2(h) = k^{*^{-1}} k^* \delta_1(h')$$

which contains $\delta_1(h')$. And $\Delta = j_1 k^{*^{-1}} \bar{\delta}_2$ being a well-defined homomorphism, we see that for any α and β in $k^{*^{-1}} \bar{\delta}_2(h)$, $j_1^*(\alpha) = j_1^*(\beta)$. Therefore,

$$\Delta(h) = j_1^* k^{*^{-1}} \bar{\delta}_2(h) = j_1^* \delta_1(h') = 0.$$

The following notation is convenient. _If P is a subset of Q and if $h \in H^p(Q)$ then $h \mid P$ denotes the image of h under the natural homomorphism induced by the inclusion map of P into Q._

Theorem 1.12. (Reduction Theorem). If X is a compact Hausdorff space, if A is closed, and if $h \in H^p(X)$ such that $h \mid A = 0$, then there is an open set U containing A such that $h \mid U = 0$.

Definition 1.6. (X, A) is a _compact pair_ if and only if X is a compact Hausdorff space and A is a closed subset of X.

Theorem 1.13. (Map Excision Theorem). If (X, A) and (Y, B) are compact pairs, and if

$$f: (X, A) \rightarrow (Y, B)$$

is a closed map such that f takes $X - A$ topologically onto $Y - B$, then

$$f^* : H^p(Y, B) \cong H^p(X, A).$$

Theorem 1.14. (Homotopy Lemma). If (X, A) is a compact pair and if T is a connected space, and if for each t in T

$$\lambda_t : (X, A) \to (X \times T, A \times T)$$

is defined by $\lambda_t(x, t) = (x, t)$,
then $\lambda^*_r = \lambda^*_s$ for r and s any elements of T.

Definition 1.7. A space X is unicoherent if and only if X is connected and $X = A \cup B$ with A and B closed and connected implies $A \cap B$ is connected. X is hereditarily unicoherent if every subcontinuum[2] of X is unicoherent.

Theorem 1.15. If X is a continuum and if $H^1(X) = 0$ for $G \neq 0$, then X is unicoherent.

Definition 1.8. If X is a space, if $A \subset X$ and if h is a non-zero member of $H^p(A)$, then a closed set $F \subset A$ is a floor for h if and only if $h \mid F \neq 0$ while $h \mid F' = 0$ for any closed proper subset F' of F.

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[2] A continuum is a compact connected Hausdorff space.

Theorem 1.16. (Floor Theorem). If (X, A) is a compact pair and if h is a non-zero member of $H^p(A)$, then h has a floor. Moreover, every floor is connected.

Theorem 1.17. If (X, A) is a compact pair, if $R = A \times A \cup \Delta$ where $\Delta = \{ (x, x) \mid x \in X \}$ then $H^p(X, A) \cong H^p(X / R)$ for all $p \geq 1$.

CHAPTER II

A CHARACTERIZATION OF THE CUTPOINT ORDER ON A TREE

A tree is a continuum such that every two distinct points are separated by the omission of a third point. Let X be a tree and let z be an arbitrary, but fixed element of X. Let $Q(z)$ be the set of all such pairs (a, b) in $X \times X$ such that at least one of the following three conditions is satisfied :

(i) $a = z$,

(ii) $a = b$, or

(iii) a separates z and b in X.

It turns out that $Q(z)$ is a continuous partial order on X, and with respect to this partial order z is the unique minimal element. We shall refer to $Q(z)$, for any z in X, as a cutpoint-order [1] on the tree X. The purpose of this chapter is to give a characterization of the cutpoint order on a tree (Theorem 2.4). We also obtain a new characterization of a tree from Relation-theoretic and Cohomological view-points.

Several characterizations of a tree have been given [2], [5], [23], and [24]. Perhaps the most useful of these characterizations is

Lemma 2.1. [2], [5]. A continuum X is a tree if and only if it is locally connected and hereditarily unicoherent.

An excellent proof of this lemma may be found in Ward [24].

Definition 2.1. A space X is said to be semi-locally-connected (abbreviated s.l.c.) at a point x of X provided for any open set U in X containing x there exists an open set V containing x such that $V \subset U$ and that $X - V$ has only a finite number of components. If X is s.l.c. at each of its points, it is said to be s.l.c.

In 1953, A. D. Wallace [15] proved that one-codimensional[1] compact connected and locally connected topological semigroup with unit and zero is a tree. L. W. Anderson and L. E. Ward, Jr. in 1961 [1] modified Wallace's result by eliminating the necessity of hypothesizing a unit. More precisely, they proved that if

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[1] For the definition and properties of Codimension, see Wallace [21], or Cohen [6].

X is a compact connected, locally connected, one-codimensional topological semilattice, then X is a tree. Wallace [17] improved this result by weakening the local connectedness of X to semilocal connectedness of X. These elegant results on Topological Algebra, motivated the following Lemma which bears a Relation-theoretic analogy.

Lemma 2.2. If R is a relation on a compact Hausdorff space X with RX a one-codimensional semi-locally connected subspace, such that

(i) the relation R is closed, i.e., $R = R^*$,

(ii) $H^1(Rx) = 0$ for every x in X,

(iii) the collection $\{Rx \mid x \in X\}$ has the finite intersection property (abbreviated f.i.p.), and

(iv) $Ra \cap Rb$ is connected for each pair a, b in X,

then RS is a tree for every closed subset S of X.

The proof of Lemma 2.2. depends on the following:

Lemma 2.3. If A, M and B are disjoint non-void closed subsets of a normal space X, and if A is either compact or consisting of finitely many components such that (i) M does not separate A and B in X, and (ii) for any open set U containing M there is an open subset V of U containing M such that $X - V$ has only a finite number of components, then there exists a closed and connected subset N of X such that $N \subset X - M$ and N meets both A and B.

This lemma was first proved by G. T. Whyburn [26] for the particular case in which X was assumed to be a metric continuum and card A = card M = card $B = 1$. The non-metric case was implicit in a paper by Wallace [17] but without proof. We postpone the proof of Lemma 2.3. to the end of this chapter.

Proof of Lemma 2.2. It follows from (i) and Theorem 1.1. that RX is closed and from (iii) and (iv) that

$$RX = \bigcup \{ Rx \mid x \in X \}$$

is connected and thus RX is a continuum. Similarly RS is a continuum.

Since every subcontinuum of a tree is itself a tree and since RS is a subcontinuum of RX, it is sufficient to show that RX is a tree.

We first show $H^1(RX) = 0$. If there were a non-zero $h \in H^1(RX)$, then there would be a maximal (non-void) tower $\mathfrak{J}$ of closed subsets A of X such that $h \mid RA \neq 0$. Let $A_o = \bigcap \{ A \mid A \in \mathfrak{J} \}$. Then

$h \mid RA_o \neq 0$, for if $h \mid RA_o = 0$, then by the Reduction Theorem (Chapter 1, Theorem 1.12.) there would be an open $V \supset RA_o$ such that $h \mid V^* = 0$. It would then follow from Theorem 1.1. that $R^{[-1]}\, V$ would be an open set containing A_o. If $R^{[-1]}V$ is designated by U then $R\,U \subset V$ so that there is an A in $\mathcal{T}$ with $A \subset U$ and $RA \subset RU \subset V^*$; therefore $h \mid RA = 0$, a contradiction.

Case 1. Card $A_o = 1$, i.e., $A_o = \{ x \}$. By (ii) $H^1(RA_o) = 0$, a contradiction.

Case 2. Card $A_o > 1$. Write $A_o = A_1 \cup A_2$ where both A_1 and A_2 are proper closed subsets of A_o. We consider the following part of the Mayer-Vietoris exact sequence (Chapter I, Theorem 1.10.),

$$H^o(RA_1 \cap RA_2) \xrightarrow{\Delta} H^1(RA_o) \xrightarrow{J^*} H^1(RA_1) \times H^1(RA_2).$$

Since by (iii) and (iv)

$$RA_1 \cap RA_2 = \bigcup \{ RA \cap Rb \mid (a,\, b) \in A_1 \times A_2 \}$$ is

connected, then $\Delta = 0$ (Theorem 1.11.), and

$$h \quad RA_o \in \operatorname{Ker} J \;\; = \operatorname{Im}\Delta = 0,$$

a contradiction.

RX is a continuum and $H^1(RX) = 0$ imply that RX is unicoherent (Theorem 1.15.). RX being of codimension one and $H^1(RX) = 0$ imply that $H^1(K) = 0$ for every closed subset K of RX [21], and thus every subcontinuum of RX is unicoherent.

We now prove that every two points of RX are separated in RX by a third point. Suppose there were two points a and b such that no point separated a and b in RX. Then by Lemma 2.3., for any p different from both a and b, there would be a continuum P which would be irreducible from a to b and which would not contain p. If q were an element of P distinct from a and b there would also be a continuum Q irreducible from a to b and which would not contain q. But then $P \cup Q$ would be a subcontinuum of RX which would not be unicoherent, since $P \cap Q$ by our selection of P and Q is obviously not connected. This contradiction completes the proof.

Theorem 2.4. If X is a compact Hausdorff space, and if P is a relation on X, then the following conditions

(i) X is of 1-codimension and s.l.c.,

(ii) P is a closed partial order,

(iii) P is left monotone, i.e., Px is connected, and $H^1(Px) = 0$ for every x in X,

(iv) $\{Px \mid x \in X\}$ has the f.i.p., and

(v) P is right monotone, i.e., xP is connected for every x in X,

are necessary and sufficient conditions that X be a tree,[2] and that P be a cutpoint-order.

Proof: We first prove the sufficiency. Conditions (ii), (iv) and the first half of (iii) imply that

$$Pa \cap Pb = \cup \{ Px \mid x \in Pa \cap Pb \}$$

is connected, and thus Lemma 2.2. yields that X is a tree.[3]

Since X is compact and $\{ Px \mid x \in X \}$ has the f.i.p., then

$$\{ Px \mid x \in X \} \neq \Box \ .$$

Indeed, it is a single point, the unique P-minimal element of X. Let us denote by O the set

$$\{ Px \mid x \in X \} .$$

We prove that $P = Q(O)$. If $(a, b) \in Q(O)$ such that $a = O$ or $a = b$, then clearly (a, b) must be also in P. If a separates O and b in X, then since Pb is a continuum containing O and b, it must contain a, and we again conclude that (a, b) is in P. Thus $Q(O) \subset P$. Conversely, if (a, b) is in P, then since a is in $aP \cap Pb$, and since both aP and Pb are continua, then $aP \cup Pb$ is a subcontinuum of the tree X, and therefore by Lemma 2.1. it is unicoherent. Thus $aP \cap Pb$ is also a continuum. Now, by virtue of the Hausdorff-Maximality-Principle, $aP \cap Pb$ has a maximal P-chain, C, and

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[2] Condition (v) is not necessary for X to be a tree.

[3] Only reflexivity and transitivity of P are used.

(a) C is closed (Theorem 1.5.), and

(b) C is connected.

For if C were not connected there would be two non-void disjoint closed sets A and B such that $C = A \cup B$ and $b \in B$. The set A contains a maximal element m. Define A' and B' by the equations

$$A' = Pm \cap C \quad \text{and} \quad B' = C - Pm.$$

$B' \subset mP$, and since $A \subset A'$, then $B' \subset B$. Now

$$A' \cap B'^{*} \subset Pm \cap (mP \cap B) = (Pm \cap mP) \cap B = \square ,$$

therefore

$$C = A' \cup B'$$

is a separation. If b_o designates the minimal element in B', then by the maximality of C

$$mP \cap Pb_o = \{ m, b_o \}$$

which contradicts the connectedness of $mP \cap Pb_o$. Therefore, any maximal P-chain in $aP \cap Pb$ is connected.

(c) $aP \cap Pb$ has a unique maximal P-chain, which we denote by $C_p(a, b)$.

If C and C' were two distinct maximal P-chains in $aP \cap Pb$, then both C and C' would contain a and b, and $C \cup C'$ would then be connected, and hence $C \cap C'$ would be connected. But for $x \in C - C'$

$$C \cap C' = (Px \cap C \cap C') \cup (xP \cap C \cap C') =$$
$$C \cap (Px \cup xP) \cap C'$$

is obviously a separation, a contradiction.

Since $(0, b) \in Q(0) \subset P$ and since X is a tree, there is a unique connected Q-chain [22], [23] $C_Q(0, b) \subset Pb$ which contains both 0 and b. Pb must also have a connected P-chain containing both 0 and b and this P-chain must be unique. We denote by $C_p(0, b)$ the unique connected P-chain in Pb containing 0 and b. Since a Q-chain is also a P-chain, then

$$C_P(0, b) = C_Q(0, b).$$

Similarly, there is a unique connected P-chain $C_P(0, a)$ in Pa containing both 0 and a. It is clear that

$$C_P(0, a) \cup C_P(a, b) = C_P(0, b) = C_Q(0, b).$$

As a consequence, $a \in C_Q(0, b)$ and hence $(a, b) \in Q$ which was to be proved.

We next prove the necessity. Let X be a tree and let P be the cutpoint order on X with respect to a point z in X. We prove that X and P satisfy the conditions (i), (ii), (iii), (iv) and (v) stated in the theorem:

Proof of (i). By Ward [24] a tree is a compact connected commutative idempotent semigroup with zero, therefore it is acyclic [17]. Hence in particular

$H^1(X) = 0$. We now show $H^1(A) = 0$ for every $A = A^* \subset X$ and thus X is of one codimension, unless X is degenerate. Suppose on the contrary that $H^1(A) \neq 0$ for some closed subset A of X. If h is a non-zero member of $H^1(A)$, then by the Floor Theorem (Theorem 1.16.) there is a floor $F \subset A$ for h, which is connected. The set F being a subcontinuum of a tree is itself a tree and hence is acyclic. Therefore, $H^1(F) = 0$ which contradicts the fact that F is a floor, and thus $H^1(A) = 0$. The semilocal connectedness of X follows from the fact that X is compact and locally connected.

Proof of (ii). This is proved in Ward [24].

Proof of (iii). The cutpoint order P is order dense [23] , and since by (ii) $P = P^*$ we have that every maximal P-chain in Px is connected [22], thus Px is connected. Indeed, Px itself is a tree and therefore as has been proved in (i), $H^1(Px) = 0$.

Proof of (iv). This is obvious, since P has the least element z.

Proof of (v). Replacing Px by xP in the argument of (iii), we easily obtain the connectivity of xP.

Proof of Lemma 2.3. (1) _There is a component_ A_o _of_ A _such that_ M _does not separate_ A_o _and_ B _in_ X. For if otherwise, to each component A_λ of A there is a pair of disjoint open sets G_λ and T_λ such that

$X - M = G_\lambda \cup T_\lambda$ and $G_\lambda \supset A_\lambda$, $T_\lambda \supset B$. Since A is either compact or consisting of finitely many components, there is a finite subfamily $\{G_1, G_2, \ldots, G_m\}$ of G_λ's such that $A \subset \cup \{ G_i \mid i = 1, 2, \ldots, m \}$. Whence $G = \bigcup_{i=1}^{m} G_i$ and $T = \bigcap_{i=1}^{m} T_i$ are two disjoint open sets such that $X - M = G \cup T$ and $G \supset A$, $T \supset B$ so that the hypothesis (i) is contradicted.

Throughout the rest of the proof, let $\#[S]$ be the number of components of the space S. Let $\mathcal{V} = \{ V_\lambda \mid \lambda \in \Lambda\}$ be the collection of all open sets V_λ containing M but missing A such that $\#[X - V_\lambda]$ is finite. For each $V_\lambda \in \mathcal{V}$ let R_λ be the component of $X - V_\lambda$ that contains the component A_o of A. Designate $R = \cup \{ R_\lambda \;\; \lambda \in \Lambda \}$.

(2) R *is open*. Let $y \in R$, and let $V_\lambda \in \mathcal{V}$ be such that R_λ contains y. By the normality of X there is a $V_\mu \in \mathcal{V}$ with $V_\mu \subset V_\mu^* \subset V_\lambda$. Since $X - V_\mu$ has only a finite number of components and since

$$(X - V_\mu)^\circ \supset X - V_\mu^* \supset X - V_\lambda,$$

we have $R_\lambda \subset R_\mu^\circ \subset R$. Therefore R is open.

(3) *For each* $V_\alpha \in \mathcal{V}$, *there is an* R_λ *containing* R_α *such that*

$$\#[R_\mu \cap (X - V_\alpha)] = \#[R_\lambda \cap (X - V_\alpha)]$$

for all R_μ *containing* R_λ. Let $C_1, C_2, \ldots, C_n$ be the n components of $X - V_\alpha$, so that

$$X - V_\alpha = C_1 \cup C_2 \cup \cdots \cup C_n$$

where $C_1 = R_\alpha$, then for any V_β contained in V_α we have $X - V_\beta$ containing $X - V_\alpha$, so that each C_i must be either totally contained in R_β, the component of $X - V_\beta$ containing A_o, or disjoint from R_β. Thus

$$\#[R_\beta \cap (X - V_\alpha)] \leq \#(X - V_\alpha) = n.$$

But,

$$\#[R_\delta \cap (X - V_\alpha)] \leq \#[R_\tau \cap (X - V_\alpha)]$$

if $R_\tau \supset R_\delta \supset R_\alpha$. Hence there is an $R_\lambda \supset R_\alpha$ such that the number $\#[R_\lambda \cap (X - V_\alpha)]$ is the maximal so that

$$\#[R_\mu \cap (X - V_\alpha)] = \#[R_\lambda \cap (X - V_\alpha)]$$

for all R_μ containing R_λ.

(4) *For each* $V_\alpha \in \mathcal{U}$ *there is an* R_λ *such that*

$$R_\mu \cap (X - V_\alpha) = R_\lambda \cap (X - V_\alpha)$$

for all R_μ *containing* R_λ. By virtue of (3) above, there is an R_λ containing R_α such that

$$\#[R_\mu \cap (X - V_\alpha)] = \#[R_\lambda \cap (X - V_\alpha)]$$

for all R_μ containing R_λ. We note, further, that

$$R_\mu \cap (X - V_\alpha) = R_\lambda \cap (X - V_\alpha)$$

for all R_μ containing R_λ. For if as in (3) above, we express

$$X - V_\alpha = C_1 \cup C_2 \cup \cdots \cup C_n$$

where, without loss of generality, C_i is contained in R_λ

for $i = 1, 2, \ldots, k$ $(k \leq n)$ and $C_i \cap R_\lambda = \square$ for $i = k + 1, \ldots, n$, then it follows from R_μ containing R_λ and

$$\#[R_\mu \cap (X - V_\alpha)] = \#[R_\lambda \cap (X - V_\alpha)]$$

that $C_i \cap R_\mu = \square$ for $i = k + 1, \ldots, n$. And therefore,

$$R_\mu \cap (X - V_\alpha) = C_1 \cup C_2 \cup \ldots \cup C_k$$
$$= R_\lambda \cap (X - V_\alpha).$$

(5) $R = R^* - M$; that is, R is closed in $X - M$. If y is a point of $R^* - M$ then there exists a V_α in $\mathcal{V}$ missing y. By the normality of X there is a V_β in $\mathcal{V}$ such that $V_\beta \subset V_\beta^* \subset V_\alpha$. By (4) there is an R_λ containing R_β such that

$$R_\mu \cap (X - V_\beta) = R_\lambda \cap (X - V_\beta)$$

for all R_μ containing R_λ. Furthermore, if U is an open set about y, then $U \cap R_\lambda \neq \square$. For, if we designate the set $U \cap (X - V_\beta^*)$ by W then y is in both $R - M$ and W so that $W \cap R \neq \square$, and there is an R_δ such that $W \cap R_\delta \neq \square$. Without loss of generality we may assume that R_δ contains R_λ. Then

$$\square \neq W \cap R_\delta = W \cap R_\delta \subset (X - V_\beta)$$
$$= W \cap R_\lambda \cap (X - V_\beta)$$
$$= W \cap R_\lambda.$$

Thus, $U \cap R_\lambda \supset W \cap R_\lambda \neq \square$, so that y is in $R_\lambda = R_\lambda$. Therefore, R is closed in $X - M$.

It is to be noted that R meets B and hence $R_\tau \cap B \neq \square$ for some τ; because otherwise

$$X - M = R \cup (X - M - R)$$

would be a separation of $X - M$ between A_0 and B, so that (1) would be contradicted. We now conclude the result by taking $N = R_\tau$.

CHAPTER III

FIXED POINTS AND MINIMAL ELEMENTS

In the Symposium of General Topology and its Relations to Modern Analysis and Algebra (Prague 1961), Professor A. D. Wallace announced [18] among other things the following fixed point theorem.

Theorem 3.1. [18, Theorem 5]. If X is a continuum, if P is a closed left monotone partial order on X such that $PA^* \subset (PA)^*$ for each $A \subset X$, and if z separates Pa and Pb in X, then $Pz = z$.

Wallace applied this theorem to prove

Theorem 3.2. [18, Theorem 6]. If X is a continuum and if P is a closed left monotone partial order on A such that $PA^* \subset (PA)^*$ for each $A \subset X$, then the set K of P-minimal elements is connected.

However, in [18] Wallace gave no proof of the fixed point theorem. For the sake of completeness, we will give in this chapter a proof of Theorem 3.1 by first proving Theorem 3.2 and using this result to establish Theorem 3.1. We will also show that under certain conditions the statements

(i) If z separates Pa and Pb in X then

$Pz = z$.

(ii) The set of P-minimal elements is connected.
are equivalent (Theorem 3.4).

Furthermore, if P is a partial order on the continuum X and K is the P-minimal elements in X, we will prove in Theorems 3.5, 3.6 and 3.7 that the cohomology groups of X and those of K are isomorphic for all non-negative dimensions for certain classes of X with suitable choice of P.

The following lemma will be used in the proofs of foregoing theorems.

Lemma 3.3. [cf. 16]. If X is a compact Hausdorff space and if P is a lower semi-closed partial order on X such that $PA^* \subset (PA)^*$ for each $A \subset X$, then the set K of P-minimal elements is closed.

Proof. Suppose by way of contradiction that there is an x in $K^* - K$. Then since Px is closed and X is compact, Px has a minimal element which must be a P-minimal element in X, and thus

$$Px \cap K \neq \square .$$

Let y be any element in $Px \cap K$. Obviously $x \neq y$, so that there is an open set U containing x whose closure excludes y. If $V = U \cap K$, then $PV = V$, and x is in V^*. Since

$$PV^* \subset (PV)^* = V^*,$$

then $PV^* \subset U^*$ and hence $y \in Px \subset PV^* \subset U^*$. This contradicts the fact that $y \notin U^*$. Therefore the set K is closed.

Proof of Theorem 3.2. Let us assume contrary to the conclusion of the theorem that K is not connected. By Lemma 3.3 there are two disjoint non-void closed sets A and B such that $K = A \cup B$. Since P is closed, then AP and BP are closed and $X = AP \cup BP$. Furthermore, the connectedness of X yields $AP \cap BP \neq \Box$. Thus $AP \cap BP$ has a minimal element, say t. Since

$$Pt = (Pt \cap AP) \cup (Pt \cap BP),$$

then

$$(Pt \cap AP) \cap (Pt \cap BP) = Pt \cap (AP \cap BP) = t.$$

If we designate

$$C = (Pt \cap AP) - t \quad \text{and} \quad D = (Pt \cap BP) - t,$$

then both C and D are open in Pt. Furthermore, C and D are both non-void. For if C is void, then $Pt \cap AP = t$, and so t must be in A. This implies that t is not in BP which contradicts the fact that t is a minimal element in $AP \cap BP$. Similarly, D is not void. Consequently, $Pt - t = C \cup D$ is a separation. The connectedness of Pt yields $C^* = Pt \cap AP$ and $D^* = Pt \cap BP$. Now, for each x in $Pt - t$ we have $Px \subset Pt - t$ and the connectedness of Px then implies that either $Px \subset C$ or $Px \subset D$. Thus, for each x in C we see that $Px \subset C$, and hence $PC \subset C$, and it follows that

$$C^* \supset (PC)^* \supset PC^* = P(C \cup t) \supset Pt \supset D$$

which is a contradiction. Therefore K is connected.

Proof of Theorem 3.1. Let $X - z = A \cup B$ where A and B are non-void separated sets and $aP \subset A$ and $bP \subset B$. Let K be the set of P-minimal elements. Now, if $Pz \neq z$, then $K \subset X - z$ and

$$K = (K \cap A) \cup (K \cap B)$$

is a separation for K which contradicts the connectedness of K.

Theorem 3.4. If X is a continuum and P is a closed left monotone partial order on X such that the set of P-minimal elements K is closed, then the following statements are equivalent:

(i) If z separates Pa and Pb in X then $Pz = z$.

(ii) The set K is connected.

Proof. (i) implies (ii). Suppose the set K is the union of two disjoint non-void closed sets A and B. Since K is the set of P-minimal elements then

$$X = KP = (A \cup B)P = AP \cup BP.$$

And the hypotheses that X is connected and P is closed yield $AP \cap BP \neq \Box$. Let z be a P-minimal element of $AP \quad BP$. Then there are elements $a \in A$ and $b \in B$ such that $\{a, b\} \subset Pz - z$. The equalities

$$Pz = (Pz \cap AP) \cup (Pz \cap BP), \text{ and}$$

$$Pz \cap (AP \cap BP) = z$$

show that

$$Pz - z = (Pz \cap AP - z) \cup (Pz \cap AP - z)$$

is a separation, that is, z separates $a = Pa$ and $b = Pb$ in the continuum Pz, and so by (i) z is a minimal element which is neither in A nor B. This contradiction establishes the connected ness of K.

(ii) implies (i). The proof as given for Theorem 3.1 applies here as well.

We now present an example to motivate Theorem 3.5.

Example. Let $X = \{(a, b) \mid a^2 + b^2 = 1\} \cup \{(0, b) \mid 1 \leq b \leq 2\}$ for reals a and b. Let X be endowed with the Euclidean topology. Let

$$P = \{((0, b_1), (0, b_2)) \mid 1 \leq b_1 \leq b_2 \leq 2\} \cup \Delta$$

where $\Delta = \{(x,x) \mid x \in X\}$. Then $P = P^*$ is a both left and right monotone partial order on X such that $PA^* \subset (PA)^*$ for each $A \subset X$ and

$$K = \{(a, b) \mid a^2 + b^2 = 1\} = S^1 \quad \text{(1-sphere)}$$

is the set of P-minimal elements.

It is to be noted in this example that $H^1(Px) = 0$ for each $x \in X$ while $H^1(K) \cong H^1(S) \neq 0$ for any non-trivial coefficient group. However the following equality

$H^p(X) \cong H^p(K)$ holds for all non-negative dimensions.

It is interesting to seek conditions on X and a relation P which imply the equality $H^p(X) \cong H^p(K)$ for all non-negative dimensions. The purpose of the next theorem is to take a small step in this direction.

Theorem 3.5. If X is a continuum, and if P is an upper semiclosed partial order on X such that

(i) $PA^* \subset (PA)^*$ for all $A \subset X$,

(ii) there exists an element u in X such that $X - K \subset Pu$ where $K = \{ x \mid Px = x \}$, and

(iii) the quotient space $X / K \times K \cup \Delta$ modulo $K \times K \cup \Delta$ is a topological semilattice under the natural partial order induced by P, where $\Delta = \{ (x, x) \mid x \in X \}$, then, $H^p(X) \cong H^p(K)$ for all integers $p \geq 0$.

Proof. For simplicity in notation we write $Y = X / K \times K \cup \Delta$. Since every compact topological semilattice has a (unique) zero, we write z for the zero of the semilattice Y. By the hypothesis (ii), Y has a (unique) unit which will be denoted by u also since no confusion is likely to occur.

We will accomplish the proof in three steps.

(a) $H^p(X, K) \cong H^p(Y, z)$ for all integers $p \geq 0$. Let $f : (X, K) \to (Y, z)$ be the natural map. Since (X, K) is a compact pair, f is a closed map, and moreover, f

takes $X - K$ topologically onto $Y - z$. Thus by the Map Excision Theorem (Theorem 1.13) $f^* : H^p(Y, z) \to H^p(X, K)$ is an isomorphism.

(b) $H^p(X, K) = 0$ for all integers $p \geq 0$. We establish this by showing $H^p(Y, z) = 0$ for all integers $p \geq 0$. Define $f_t : (Y,z) \to (Y,z)$ by $f_t(y) = t \wedge y$ for all $t \in Y$, where $\wedge$ is the semilattice operation on Y. If for each $t \in Y$ define $\lambda_t : (Y, z) \to (Y \times Y, z \times Y)$ by $\lambda_t(y) = (y, t)$, then $f_t = \wedge \circ \lambda_t$. Since by the Homotopy Lemma (Theorem 1.14) $\lambda_z^* = \lambda_u^*$, thus

$$f_z^* = \lambda_z^* \circ \wedge^* = \lambda_u^* \circ \wedge^* = f_u^*.$$

Since f_u^* is the identity map, so is f_z^*. Let $i : (z,z) \subset (Y, z)$ be an inclusion map and let $h : (Y, z) \to (z, z)$ be defined by $h(y) = z$ for all $y \in Y$. Since $f_z = i \circ h$, the following diagram

$$\begin{tikzcd}
H^p(Y, z) \arrow[rr, "f_u^* = f_z^*"] \arrow[dr, "i^*"'] & & H^p(Y, z) \arrow[dl, "h^*"] \\
& H^p(z, z) &
\end{tikzcd}$$

is analytic, for all non-negative integers p, that is $f_u^* = f_z^* = h^* \circ i^*$. Since $H^p(z, z) = 0$ for all integers $p \geq 0$ and $H^p(Y, z) = f_u^*(H^p(Y, z)) \subset h^*(H^p(z, z))$ for all integers $p \geq 0$, then $H^p(Y, z) = 0$ for all integers $p \geq 0$ as desired.

(c) $H^p(X) \cong H^p(K)$ *for all integers* $p \geq 0$. By combining the above results with the exact sequence for the triple $(X, K, \square)$ (Theorem 1.7), we obtain the exact sequence

$$0 = H^p(X, K) \to H^p(X) \to H^p(K) \to H^{p+1}(X, K) = 0$$

for $p = 0, 1, 2, \ldots$. Consequently, $H^p(X)$ and $H^p(K)$ are isomorphic for all non-negative dimensions.

We now center our attention on the case in which X is of codimension one.

Theorem 3.6. If P is a closed partial order on a continuum X and if $\pi : X \to X / K \times K \cup \Delta$ designates the natural map, such that

(i) $PA^* \subset (PA)^*$ for all $A \subset X$,

(ii) P is left monotone and $H^1(\pi(Px)) = 0$ for all $x \in X$,

(iii) X is of codimension one,

then $H^p(X) \cong H^p(K)$ for all $p \geq 0$.

Proof. Denote $\tilde{X} = X / K \times K \cup \Delta$ and

$$\tilde{P} = \{ (\pi(x), \pi(y)) \mid (x, y) \in P \}$$

then since X is compact and π continuous, $\tilde{P}$ is closed and left monotone; indeed we have $\tilde{P}\pi(x) = \pi(Px)$. By virtue of Theorem 1.17, we have $H^p(X, K) \cong H^p(\tilde{X})$ for all

$p \geq 1$. Since X is of codimension one we have $H^p(X, K) = 0$ for $p \geq 2$ [21]. We show $H^1(X, K) = 0$ by proving $H^1(X) = 0$. Using the same argument employed in the proof of Lemma 2.2 one sees that $H^1(\tilde{P}S) = 0$ for all closed sets S in X; in particular we have $H^1(\tilde{X}) = 0$. We have noted in part (c) of the proof for Theorem 3.5 that if $H^p(X, K) = 0$ for all $p \geq 1$ then $H^p(X) \cong H^p(K)$ for all $p \geq 1$. The equality $H^0(X) = H^0(K)$ follows from the fact that both X and K are connected [Theorem 3.2].

We remark that the hypothesis (iii) in Theorem 3.6 may be weakened to :

(iii) $\operatorname{cd}(X - K)^* = 1$,

and this may even be dropped completely if each Px is a chain, as will be seen in the following

Theorem 3.7. If P is a closed partial order on a continuum X such that

(i) $PA^* \subset (PA)^*$, and

(ii) each Px is a connected chain,

then $H^p(X) \cong H^p(K)$ for all $p \geq 0$.

Proof. Let $\tilde{X}$, π and $\tilde{P}$ be defined as in Theorem 3.6. Since $\tilde{P}$ is closed and πPx is a connected chain $\tilde{P}$ is closed and $\tilde{P}\pi(x)$ is a connected chain. It follows then that each $\tilde{P}\pi(x)$ is a generalized arc and hence $H^p(P\pi(x)) = 0$ for all $p \geq 1$. As it has been noted in the proof of Theorem 3.6, $H^1(\tilde{P}S) = 0$ for all closed sets S in $\tilde{X}$. We now show $H^p(\tilde{X}) = 0$ for all $p \geq 1$ by proving $H^p(\tilde{P}S) = 0$ for all $p \geq 1$ and for each closed set S in $\tilde{X}$. If there were a least integer n such that $H^n(\tilde{P}A) \neq 0$ for some closed set A in $\tilde{X}$, then n would have to be greater than 1, and $A \neq \square$. Let h be a non-zero member of $H^n(\tilde{P}A)$, then $h \mid \tilde{P}a = 0$ for each $a \in A$ and hence by the Reduction Theorem there is an open set V containing $\tilde{P}a$ such that $h \mid V^* = 0$; then by Theorem 1.1 there is an open set U containing a such that $\tilde{P}U \subset V$. Thus the collection $\mathcal{A}$ of all open subsets U of A such that $h \mid \tilde{P}U^* = 0$ forms an open cover of A. Also $\mathcal{A}$ is closed under finite union, for if U_1 and U_2 are in $\mathcal{A}$ denote $h_o = h \mid \tilde{P}U_1^* \cup PU_2^*$. Since $\tilde{P}(U_1 \cup U_2)^* = \tilde{P}(U_1^* \cup U_2^*)$ $= \tilde{P}U_1^* \cup \tilde{P}U_2^*$, to show $\mathcal{A}$ is closed under finite union it suffices to show $h_o = 0$. In the following part of Mayer-Vietoris exact sequence :

$$H^{n-1}(\tilde{P}U_1^* \cap \tilde{P}U_2^*) \xrightarrow{\Delta} H^n(\tilde{P}U_1^* \cup \tilde{P}U_2^*) \xrightarrow{J^*} H^n(\tilde{P}U_1^*) \times H^n(\tilde{P}U_2^*)$$

since $J^*(h_o) = (h \mid \tilde{P}U_1^*, h \mid \tilde{P}U_2^*) = (0, 0)$, h_o is in the image of Δ. But $H^{n-1}(\tilde{P}U_1^* \cap \tilde{P}U_2^*) = H^{n-1}(\tilde{P}S)$ where $S = \tilde{P}U_1^* \cap \tilde{P}U_2^*$ is a closed subset of $\tilde{X}$, and hence $H^{n-1}(\tilde{P}S) = 0$ by the minimality of n. This proves $h_o = 0$ so that $\mathcal{A}$ is closed under finite union. Since A is compact, A is a union of some finitely many elements of $\mathcal{A}$. Thus A must be in $\mathcal{A}$ and so $h = 0$, a contradiction. This together with Theorem 1.17 implies $H^p(X, K) = H^p(\tilde{X}) = 0$ for all $p \geq 1$. It then follows from the exact sequence for the triple $(X, K, \square)$ [Theorem 1.7] and the connectedness of X and K [Theorem 3.2] that $H^p(X) \cong H^p(K)$ for all $p \geq 0$.

Remark. Theorem 3.7 may be stated more generally by repacing the hypothesis (ii) by

(ii′) each $\pi(Px)$ is acyclic.

We conclude this chapter by exhibiting an example which answers a question in Topological Semigroups. A *clan* is a compact connected topological semigroup with unit [20]. Let S be a clan and let

$$R = \{ (x, y) \mid (x, y) \in S \times S,\ xS \cup Sx \subset yS \cup Sy \}$$

then R is a closed quasi-order on S and the set of R-minimal elements is also the minimal ideal of S. The question to be answered is: *if a clan S is a tree, is its minimal ideal an arc or a point* ?

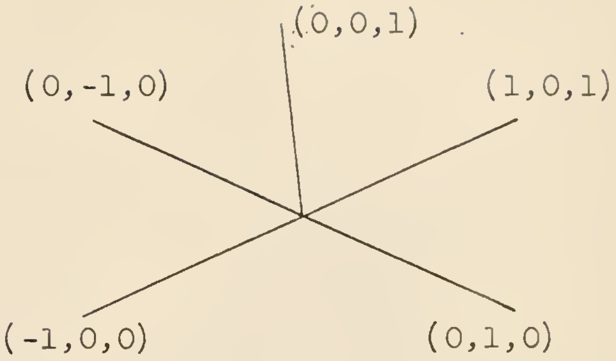
The answer is affirmative if S is abelian (or *normal*: $xS = Sx$ for all $x \in S$), but it is negative in general.

Example. Let $S = X \cup I$ be the subset of Euclidean 3-space such that

$$X = \{ (x,0,0) \mid -1 \leq x \leq 1 \} \cup \{ (0,y,0) \mid -1 \leq y \leq 1 \}$$

$$I = \{ (0,0,t) \mid 0 \leq t \leq 1 \} ;$$

let S be endowed with the Euclidean topology and let the multiplication o on S be defined below (the usual multiplication of reals is denoted by juxtaposition) :



i. $\lambda o \mu = \lambda$ for all $\lambda \in X$ and for all $\mu \in S$,

ii. (I, o) is the usual semigroup of the real unit interval; i.e.,

$$(0,0,t_1)o(0;0,t_2) = (0,0,t_1t_2),$$

iii. For each $(0,0,t) \in I$ and for any $(0,y,0) \in X$, $(0,0,t)o(x,0,0) = (tx,0,0)$ and $(0,0,t)o(0,y,0) = (0,ty,0)$.

Then, (1) (S, o) is associative.

(2) $o : S \quad S \to S$ is continuous.

(3) (S, o) is a clan with $(0,0,1)$ as unit.

(4) The minimal ideal of (S, o) is X which is neither an arc nor a point.

It is interesting to observe that the semigroup S given in the above example may be realized as a semigroup of matrices by the following one-to-one correspondence :

$$(x,y,t) \longleftrightarrow \begin{bmatrix} t & x + y\sqrt{-1} \\ 0 & 1 \end{bmatrix}$$

where $-1 \leq x, y \leq 1$, $0 \leq t \leq 1$ and $(tx)^2 + (xy)^2 + (yt)^2 = 0$. The correspondence is indeed an isomorphism. From this the assertions (1), (2) and (3) in the example are self-evident.

CHAPTER IV

FIXED POINT PROPERTIES AND INVERSE LIMIT SPACES

A space X is said to have the f. p. p. (fixed point property) if, for every continuous function f: X → X there exists some x in X such that x = f(x). Hamilton [8] has proved that the chainable metric continua have the f. p. p.

A space X is said to have the F. p. p. (fixed point property for multifunctions) if every continuous multifunction[1] $F : X \rightarrow X$ has a fixed point, i.e., there exists a point x in S such that $x \in F(x)$. Obviously if X has the F. p. p. then it has the f. p. p., but the converse need not be true. Strother [13] has exhibited two continua X and Y both have the F. p. p. but their Cartesian product $X \times Y$ fails to have the F. p. p. Borsuk [3] has constructed a decreasing sequence of three-cells whose intersection does not have the f. p. p. The intersection is the inverse limit, the bonding maps being injections. As a counter theorem to the results of Strother and of Borsuk, we prove that if $(X_\lambda, \pi_{\lambda\mu}, \Lambda)$ is an

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[1] Following Strother [12], a multifunction $F : X \rightarrow Y$ is continuous if, and only if, F(x) is closed for each x, and $F^{-1}(A)$ is open (closed) if A is open (closed).

inverse system of compact spaces such that each X_λ *has the* F. p. p. *then the inverse limit space has the* F. p. p. As a corollary to this, we obtain Ward's generalization [25] of the Hamilton theorem [8] that every chainable metric continuum has the F. p. p. Our result is, indeed, stronger than the Ward's, since it includes some of the non-metric chainable continua as well.

Definition 4.1. The collection $(X_\lambda, \pi_{\lambda\mu}, \Lambda)$ is as *inverse system of spaces* if:

(i) Λ is a directed set,

(ii) λ in Λ implies that X_λ is a Hausdorff space,

(iii) whenever $\lambda > \mu$ there is a continuous function $\pi_{\lambda\mu} : X_\lambda \to X_\mu$,

(iv) if $\lambda > \mu$ and $\mu > \nu$, then $\pi_{\lambda\nu} = \pi_{\mu\nu}\pi_{\lambda\mu}$.

The function $\pi_{\lambda\mu}$ is called a *bonding map*. If λ is in Λ let S_λ be the subset of the Cartesian product $P\{X_\lambda \mid \lambda \in \Lambda\}$, defined by

$$S_\lambda = \{x \mid \text{if } \lambda > \mu \text{ then } \pi_{\lambda\mu}x(\lambda) = x(\mu)\} ,$$

where $x(\lambda)$ denotes the λ-th coordinate of x.

Definition 4.2. The *inverse limit space* X_∞ of the inverse system of spaces $(X_\lambda, \pi_{\lambda\mu}, \Lambda)$ is defined to be

$$X_\infty = \bigcap\{S_\lambda \mid \lambda \in \Lambda\}$$

endowed with the relative topology inherited from the product topology for $P\{X_\lambda \mid \lambda \in \Lambda\}$; in notation $X = \underleftarrow{\lim} (X_\lambda, \pi_{\lambda\mu}, \Lambda)$.

We write $p_\lambda : P\{X_\lambda \mid \lambda \in \Lambda\} \to X_\lambda$ for the λ-th projection of $P\{X_\lambda \mid \lambda \in \Lambda\}$, i.e., $p_\lambda(x) = x(\lambda)$ for all x in $P\{X_\lambda \mid \lambda \in \Lambda\}$; the restriction $p_\lambda \mid X_\infty$ will be denoted by π_λ which will be called a projection map. It is readily seen from the definition that an element x of $P\{X_\lambda \mid \lambda \in \Lambda\}$ is in X_∞ if and only if $\pi_{\lambda\mu}\pi_\lambda(x) = \pi_\mu(x)$ whenever $\lambda > \mu$. A more detailed account of inverse limit space may be found in Lefschetz [10], Eilenberg and Steenrod [7], Capel [4] and Mardešić [11].

The following known results (see, e.g., [4], [10]) will be used.

Lemma 4.3. (i) The collection $\{\pi_\lambda^{-1}(U_\lambda) \mid \lambda \in \Lambda$ and U_λ is an open subset of $X_\lambda\}$ forms a basis for the topology of X_∞.

(ii) The inverse limit space X_∞ is Hausdorff; if $\lambda \in \Lambda$, S_λ is a closed subset of $P\{X_\lambda \mid \lambda \in \Lambda\}$ so that X_∞ is closed in $P\{X_\lambda \mid \lambda \in \Lambda\}$.

(iii) If X_λ is compact for each λ in Λ then X_∞ is compact; if, in addition, each X_λ is non-void then X_∞ is non-void.

(iv) If X_λ is a continuum for each $\lambda \in \Lambda$ then the inverse limit space is a continuum.

Lemma 4.4. If A is a compact subset of X_∞ and if $\pi'_{\lambda\mu} = \pi_{\lambda\mu} \mid \pi_\lambda(A)$, then $(\pi_\lambda(A), \pi'_{\lambda\mu}, \Lambda)$ is an inverse system of spaces such that $A = \underleftarrow{\lim} (\pi_\lambda(A), \pi'_{\lambda\mu}, \Lambda)$ and each bonding map $\pi_{\lambda\mu}$ is onto.

In the sequel, since we are only interested in compact spaces, each projection map π_λ will be assumed to be onto; for if otherwise, by virtue of Lemma 4.4, each X_λ may be replaced by $\pi_\lambda(X_\infty)$ without disturbing the resulting inverse limit space. We are now ready to state our main result.

Theorem 4.5. Let $(X_\lambda, \pi_{\lambda\mu}, \Lambda)$ be an inverse system of compact spaces such that each X_λ has the F. p. p., then the inverse limit space X_∞ also has the F. p. p.

We divide the proof of this theorem into the following steps. In Lemmas 4.6, 4.7 and 4.8 X_∞ will be assumed to be the inverse limit space of the inverse system $(X_\lambda, \pi_{\lambda\mu}, \Lambda)$ of compact spaces.

Lemma 4.6. If $F : X_\infty \to X_\infty$ is a continuous multifunction, define $F_\lambda : X_\lambda \to X_\lambda$ by $F_\lambda = \pi_\lambda F \pi_\lambda^{-1}$ for each λ. Then F_λ is a continuous multifunction.

Proof. (i) If t is in X_λ, then since $F : X_\infty \to X_\infty$ is a continuous multifunction, by Theorem 1.1 and Theorem 1.4 of Chapter I, $F\pi_\lambda^{-1}(t)$ is a closed subset of X_λ. Thus,

$F_\lambda(t)$ is closed for every t in X_λ.

(ii) If C_λ is a closed subset of X_λ, then $F_\lambda^{-1}(C_\lambda)$ is closed. It is readily seen that $F^{-1}\pi_\lambda^{-1}(C_\lambda)$ is closed in X_∞ and hence compact; therefore $\pi_\lambda F^{-1}\pi_\lambda^{-1}(C_\lambda) = F_\lambda^{-1}(C_\lambda)$ is compact and hence closed.

(iii) If U_λ is open in X_λ, then $F_\lambda^{-1}(U_\lambda)$ is open. π_λ and F being continuous, $F^{-1}\pi_\lambda^{-1}(U_\lambda)$ is an open subset of X_∞. It follows then, by virtue of Lemma 4.3 (i), that $\pi_\lambda F^{-1}\pi_\lambda^{-1}(U_\lambda) = F_\lambda^{-1}(U_\lambda)$ is open.

Thus, by (i), (ii) and (iii) above, $F_\lambda : X_\lambda \to X_\lambda$ is continuous.

Lemma 4.7. Let $F : X_\infty \to X_\infty$ be a continuous multifunction, let $F_\lambda : X_\lambda \to X_\lambda$ be defined as in Lemma 4.6. Then, for each x in X_∞,

(i) $(F_\lambda \pi_\lambda(x), \pi_{\lambda\mu}, \Lambda)^2$ and $(\pi_\lambda F(x), \pi_{\lambda\mu}, \Lambda)$ are inverse systems of compact spaces,

(ii) $\varprojlim (F_\lambda \pi_\lambda(x), \pi_{\lambda\mu}, \Lambda) = \varprojlim (\pi_\lambda F(x), \pi_{\lambda\mu}, \Lambda)$,

(iii) $F(x) = \varprojlim (F_\lambda \pi_\lambda(x), \pi_{\lambda\mu}, \Lambda)$.

Proof. Since each F_λ is continuous (Lemma 4.6) and each X_λ is compact, so is $F_\lambda \pi_\lambda(x)$ for all $\lambda \in \Lambda$.

---

[2] For simplicity in symbolism, henceforth if $A \subset \varprojlim (X_\lambda, \pi_{\lambda\mu}, \Lambda)$ then $(\pi_\lambda A, \pi_{\lambda\mu}, \Lambda)$ will mean $(\pi_\lambda A, \pi_{\lambda\mu}|\pi_\lambda A, \Lambda)$.

To show that $(F_\lambda \pi_\lambda(x), \pi_{\lambda\mu}, \Lambda)$ forms an inverse system, it suffices to show $\pi_{\lambda\mu} F_\lambda \pi_\lambda(x) \subset F_\mu \pi_\mu(x)$ whenever $\lambda > \mu$. To this end we first observe

$$\pi_\lambda(x) \subset (\pi_{\lambda\mu}^{-1}\pi_{\lambda\mu})\pi_\lambda(x) = \pi_{\lambda\mu}^{-1}\pi_\mu(x),$$

since $\pi_{\lambda\mu}\pi_\lambda = \pi_\mu$. From this we have

$$\begin{aligned} \pi_{\lambda\mu}F_\lambda\pi_\lambda(x) &\subset \pi_{\lambda\mu}F_\lambda\pi_{\lambda\mu}^{-1}\pi_\mu(x) \\ &= \pi_{\lambda\mu}(\pi_\lambda F\ \pi_\lambda^{-1})\pi_{\lambda\mu}^{-1}\pi_\mu(x) \\ &= (\pi_{\lambda\mu}\pi_\lambda)F(\pi_{\lambda\mu}\pi_\lambda)^{-1}\pi_\mu(x) \\ &= \pi_\mu F\ \pi_\mu^{-1}\pi_\mu(x) \\ &= F_\mu\pi_\mu(x), \end{aligned}$$

by the definition of F_λ, F_μ and the equality $\pi_{\lambda\mu}\pi_\lambda = \pi_\mu$.

The fact that $(\pi_\lambda F(x), \pi_{\lambda\mu}, \Lambda)$ forms an inverse system follows from Lemma 4.4.

(ii) For each $\lambda \in \Lambda$ and any $x \in X_\infty$, we have $x \in \pi_\lambda^{-1}\pi_\lambda(x)$ and hence,

$$\pi_\lambda F(x) \subset \pi_\lambda F\ \pi_\lambda^{-1}\pi_\lambda(x) = (\pi_\lambda F\ \pi_\lambda^{-1})\pi_\lambda(x) = F_\lambda\pi_\lambda(x).$$

Thus,

$$\underleftarrow{\lim}\ (\pi_\lambda F(x), \pi_{\lambda\mu}, \Lambda) \subset \underleftarrow{\lim}\ (F_\lambda\pi_\lambda(x), \pi_{\lambda\mu}, \Lambda).$$

To prove the other inclusion, we show

$$X_\infty - \underleftarrow{\lim}\ (\pi_\lambda F(x), \pi_{\lambda\mu}, \Lambda) \subset X_\infty - \underleftarrow{\lim}\ (F_\lambda\pi_\lambda(x), \pi_{\lambda\mu}, \Lambda).$$

Let y be in $X_\infty - \underleftarrow{\lim}\ (\pi_\lambda F(x), \pi_{\lambda\mu}, \Lambda)$ then, by Lemma 4.4

there exists a $\mu \in \Lambda$ such that $\pi_\mu(y) \notin \pi_\mu F(x)$. Let U_μ and V_μ be two disjoint open sets in X_μ such that

$$\pi_\mu(y) \in U_\mu \quad \text{and} \quad \pi_\mu F(x) \subset V_\mu$$

so that

$$F(x) \subset \pi_\mu^{-1} V_\mu .$$

It follows then from Lemma 4.3 (i) and the continuity of F that there exists a $\delta \in \Lambda$ and an open set U_δ in X_δ such that $x \in \pi_\delta^{-1} U_\delta$, and

(*) $$F(\pi_\delta^{-1} U_\delta) \subset \pi_\mu^{-1} V_\mu .$$

Since Λ is directed, there is a $\lambda_o \in \Lambda$ such that $\lambda_o > \mu$ and $\lambda_o > \delta$, we shall use this λ_o throughout the proof of this lemma. If we write $U_{\lambda_o} = \pi_{\lambda_o \delta}^{-1} U_\delta$ and use the equality $\pi_\delta^{-1} = \pi_{\lambda_o}^{-1} \pi_{\lambda_o \delta}^{-1}$, then (*) may be rewritten as

$$F(\pi_{\lambda_o}^{-1} U_{\lambda_o}) \subset \pi_\mu^{-1} V_\mu ,$$

and hence

$$F_{\lambda_o} U_{\lambda_o} = \pi_{\lambda_o} F \pi_{\lambda_o}^{-1}(U_{\lambda_o}) \subset \pi_{\lambda_o} \pi_\mu^{-1} V_\mu = \pi_{\lambda_o}(\pi_{\lambda_o \mu} \pi_{\lambda_o})^{-1} V_\mu$$

$$= \pi_{\lambda_o \mu}^{-1} V_\mu .$$

In particular,

$$F_{\lambda_o} \pi_{\lambda_o}(x) \subset \pi_{\lambda_o \mu}^{-1} V_\mu .$$

Similarly, one obtains

$$\pi_{\lambda_o}(y) \in \pi_{\lambda_o \mu}^{-1} U_\mu ,$$

since $\pi_\mu(y) \in U_\mu$ and $\pi_\mu = \pi_{\lambda_o \mu} \pi_{\lambda_o}$. The fact that U_μ and V_μ are disjoint implies $\pi_{\lambda_o \mu}^{-1} V_\mu \cap \pi_{\lambda_o \mu}^{-1} U_\mu = \square$ and consequently $\pi_{\lambda_0}(y) \notin F_{\lambda_0}\pi_{\lambda_0}(x)$. From this we conclude $y \notin \lim (F_\lambda \pi_\lambda(x), \pi_{\lambda\mu}, \Lambda)$ as desired.

(iii) This follows immediately from (ii) and Lemma 4.4.

Lemma 4.8. Let $F : X_\infty \to X_\infty$ be a continuous multifunction, let $F_\lambda : X_\lambda \to X_\lambda$ be defined as in Lemma 4.6 Let $E_\lambda = \{e_\lambda \mid e_\lambda \in X_\lambda \text{ and } e_\lambda \in F_\lambda(e_\lambda)\}$ then $(E_\lambda, \pi_{\lambda\mu}, \Lambda)$ forms an inverse system.

Proof. It suffices to prove $\pi_{\lambda\mu}E_\lambda \subset E_\mu$ whenever $\lambda > \mu$. Let $e_\lambda \in F_\lambda(e_\lambda)$, then

$$\begin{aligned}\pi_{\lambda\mu}(e_\lambda) &\in \pi_{\lambda\mu}F_\lambda(e_\lambda) = \pi_{\lambda\mu}\pi_\lambda F \pi_\lambda^{-1}(e_\lambda) = \pi_\mu F \pi_\lambda^{-1}(e_\lambda) \\ &\subset \pi_\mu F(\pi_\mu^{-1}\pi_\mu)\pi_\lambda^{-1}(e_\lambda) = F_\mu \pi_\mu \pi_\lambda^{-1}(e_\lambda) \\ &= F_\mu(\pi_{\lambda\mu}\pi_\lambda)\pi_\lambda^{-1}(e_\lambda) = F_\mu(\pi_{\lambda\mu}(e_\lambda)).\end{aligned}$$

Thus, $\pi_{\lambda\mu}E_\lambda \subset E_\mu$ as is to be proven.

Proof of Theorem 4.5. Since each X_λ has the F. p. p. and by Lemma 4.6 each $F_\lambda : X_\lambda \to X_\lambda$ is continuous, each E_λ is closed and non-void. By Lemma 4.8, $(E_\lambda, \pi_{\lambda\mu}, \Lambda)$ is an inverse system of compact spaces, so it has a non-void inverse limit space $\varprojlim (E_\lambda, \pi_{\lambda\mu}, \Lambda)$. We now conclude the proof by showing that each x in $\varprojlim (E_\lambda, \pi_{\lambda\mu}, \Lambda)$ is a fixed point under F; i.e., $x \in F(x)$.

If x is in $\varprojlim (E_\lambda, \pi_{\lambda\mu}, \Lambda)$ then $\pi_\lambda(x) \in E_\lambda$ for all $\lambda \in \Lambda$; i.e., $\pi_\lambda(x) \in F_\lambda \pi_\lambda(x)$ for all $\lambda \in \Lambda$. Consequently, by Lemmas 4.4 and 4.7, we have

$$x = \varprojlim (\pi_\lambda(x), \pi_{\lambda\mu}, \Lambda) \in \varprojlim (F_\lambda \pi_\lambda(x), \pi_{\lambda\mu}, \Lambda)$$
$$= F(x).$$

In fact, with the assumption of Theorem 4.5 and the notation of Lemma 4.8 together with the notation $E = \{ x \mid x \in F(x) \}$, we can make the following sharper assertion.

Theorem 4.5′. $E = \varprojlim (E_\lambda, \pi_{\lambda\mu}, \Lambda)$.

Proof. From the proof of Theorem 4.5, we have

$$E \supset \varprojlim (E_\lambda, \pi_{\lambda\mu}, \Lambda).$$

It remains to prove that

$$E \subset \varprojlim (E_\lambda, \pi_{\lambda\mu}, \Lambda) .$$

Let x be in E, then $x \in F(x)$ and therefore, for all $\lambda \in \Lambda$,

$$\pi_\lambda(x) \in \pi_\lambda F(x) \subset \pi_\lambda F(\pi_\lambda^{-1} \pi_\lambda)(x) = F_\lambda(\pi_\lambda(x)).$$

That is, $\pi_\lambda(x) \in E_\lambda$ for all λ; consequently, by Lemma 4.4 $E \subset \varprojlim (E_\lambda, \pi_{\lambda\mu}, \Lambda)$.

A chain $(U_1, U_2, \ldots, U_n)$ is a finite sequence of sets U_i such that $U_i \cap U_j \neq \square$ if and only if $| i - j | \leq 1$. A Hausdorff space X is said to be chainable if to each open cover $\mathcal{V}$ of X there is a finite open cover $\mathcal{U} = (U_1, U_2, \ldots, U_n)$ such that (i) $\mathcal{U}$ refines $\mathcal{V}$; (ii) $\mathcal{U} = (U_1, U_2, \ldots, U_n)$ forms a chain. It follows that a chainable space is a continuum. It is implicit in a paper by Isbell [9] that each metrizable chainable continuum is the inverse limit space of a sequence of (real) arcs. This together with a theorem of Strother [13] that a bounded closed interval of the real numbers has the F. p. p. imply the following result of Ward [25] as a consequence of Theorem 4.5.

Corollary 4.9. Each chainable metric continuum has the F. p. p.

Examples of inverse limit spaces of inverse systems of real arcs exist which are not metrizable; for instance, the "long line" is one such. Thus, Theorem 4.5 is a proper generalization of that of Ward's [25].

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BIOGRAPHICAL SKETCH

Shwu-yeng Tzeng Lin was born May 11, 1934 at Tainan, Formosa. In June, 1953, she was graduated from Tainan Girl's High School. In June, 1958, she received the degree of Bachelor of Sciences from Taiwan Normal University. She worked as a Research Assistant at the Mathematics Institute of Academia Sinica from 1958 to 1960. In the fall of 1960 she enrolled, and worked as a teaching assistant, in the Department of Mathematics, Graduate School of the Tulane University, New Orleans, Louisiana, and received the degree of Master of Sciences in May, 1963. From September, 1963, until the present time she has pursued her work toward the degree of Doctor of Philosophy at the University of Florida.

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This dissertation was prepared under the direction of the chairman of the candidate's supervisory committee and has been approved by all members of that committee. It was submitted to the Dean of the College of Arts and Sciences and to the Graduate Council, and was approved as partial fulfillment of the requirements for the degree of Doctor of Philosophy.

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